SOME CRITERIA FOR UNIVALENCE OF A CERTAIN INTEGRAL OPERATOR
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Abstract. The main objective of this paper is to obtain new conditions for the integral operator $F_{\alpha, \beta}(z)$ to be univalent in the open unit disk $U$. This integral operator $F_{\alpha, \beta}(z)$ was considered in a recent work \cite{3}. A number of known or new univalence conditions are shown to follow upon specializing the parameters involved in our main results.

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1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1 \}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers. We denote by $P$ the class of the functions $p(z)$ which are analytic in $U$ and satisfy the following conditions:

$$p(0) = 1 \quad \text{and} \quad \Re\{p(z)\} > 0, \quad z \in U.$$

Let $S$ denote the subclass of $A$ consisting of functions $f(z)$ which are univalent in $U$. Suppose also that $S^*$ denotes the subclass of $S$ consisting of all functions $f(z)$ in $S$ which are starlike in $U$.

The following univalence condition was derived by Ozaki and Nunokawa \cite{2}.

\begin{footnotesize}
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\end{footnotesize}
Theorem 1.1 (see [2]). Let the function \( f \in \mathcal{A} \) satisfy the following inequality:

\[
\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq |z|^2, \quad z \in \mathbb{U}.
\]  

Then \( f(z) \) is in the univalent function class \( \mathcal{S} \) in \( \mathbb{U} \).

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [6] and the references cited therein). In our present investigation we study the univalence conditions for the following integral operator:

\[
F_{\alpha, \beta}(z) := \left( \beta \int_0^z t^{\beta - \alpha - 1} [f(t)]^\alpha g(t) dt \right)^{\frac{1}{\beta}}
\]

\( (\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\}, f \in \mathcal{A}, g \in \mathcal{P}) \).

In the proof of our main result (Theorem 2.1 below), we need each of the following univalence criteria. The first univalence criterion, which is asserted by Theorem 1.2 below, is a generalization of the Ozaki-Nunokawa criterion (1.1); it was obtained by Răducanu et al. [5]. The second univalence criterion, which is asserted by Theorem 1.3 below, is a generalization of Ahlfors’s and Becker’s univalence criterion; it was proven by Pescar [3].

Theorem 1.2 (see [5]). Let \( f \in \mathcal{A} \) and \( m > 0 \) be so constrained that

\[
\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| - m - \frac{1}{2} \left| z^{m+1} \right| \leq \frac{m + 1}{2} \left| z^{m+1} \right|, \quad z \in \mathbb{U}.
\]

Then the function \( f(z) \) is analytic and univalent in \( \mathbb{U} \).

Theorem 1.3 (see [3]). Let the parameters \( \beta \in \mathbb{C} \) and \( c \in \mathbb{C} \) be so constrained that

\( \Re(\beta) > 0 \) and \( |c| \leq 1, \quad c \neq -1 \).

If \( f \in \mathcal{A} \) satisfies the following inequality:

\[
\left| c |z|^{2\beta} + \left( 1 - |z|^{2\beta} \right) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1, \quad z \in \mathbb{U},
\]

then the function \( F_\beta(z) \) given by

\[
F_\beta(z) = \left( \beta \int_0^z t^{\beta - 1} f'(t) dt \right)^{\frac{1}{\beta}} = z + \cdots
\]

is analytic and univalent in \( \mathbb{U} \).

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [11]).
**Lemma 1.4** (General Schwarz Lemma (see [1])). Let the function $f(z)$ be regular in the disk

$$U_R = \{ z : z \in \mathbb{C} \text{ and } |z| < R, \quad R > 0 \}$$

with

$$|f(z)| < M, \quad z \in \mathbb{C}, \quad M > 0$$

for a fixed number $M > 0$. If the function $f(z)$ has one zero with multiplicity order bigger than a positive integer $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R.$$

The equality in (1.6) holds true only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where $\theta$ is a real constant.

# 2. The Main Univalence Criterion

Our main univalence criterion for the integral operator $F_{\alpha,\beta}(z)$ defined by (1.2) is asserted by Theorem 2.1 below.

**Theorem 2.1.** Let the function $f \in A$ satisfy the hypothesis (1.3) of Theorem 1.2. Suppose that $M, N$ are real positive numbers, $m > 0$ and $g \in P$. Also let

$$\Re(\beta) \geq [|\alpha| ((m + 1) M + 1)) + N], \quad \alpha, \beta \in \mathbb{C}.$$

If

$$|f(z)| < M, \quad z \in U, \quad \left| \frac{zg'(z)}{g(z)} \right| \leq N, \quad z \in U$$

and

$$|c| \leq 1 - \frac{1}{\Re(\beta)} |\alpha| [(m + 1)M + 1] - \frac{1}{\Re(\beta)} N, \quad c \in \mathbb{C},$$

then the function $F_{\alpha,\beta}(z)$ defined by (1.2) is analytic and univalent in $U$.

**Proof.** We begin by observing that the integral operator $F_{\alpha,\beta}(z)$ in (1.2) can be rewritten as follows:

$$F_{\alpha,\beta}(z) = \left( \beta \int_{0}^{z} t^{\beta-1} \left( \frac{f(t)}{t} \right)^{\alpha} g(t) dt \right)^{\frac{1}{\beta}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_{0}^{z} \left( \frac{f(t)}{t} \right)^{\alpha} g(t) dt, \quad f \in A, \quad g \in P.$$
The function $f$ is indeed regular in $\mathbb{U}$ and satisfies the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$ 

Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$(2.3) \quad h'(z) = \left( \frac{f(z)}{z} \right)^{\alpha} g(z)$$

and

$$(2.4) \quad h''(z) = \alpha \left( \frac{f(z)}{z} \right)^{\alpha-1} \left( \frac{zf'(z) - f(z)}{z^2} \right) g(z) + \left( \frac{f(z)}{z} \right)^{\alpha} g'(z).$$

We easily find from $(2.3)$ and $(2.4)$ that

$$(2.5) \quad \frac{zh''(z)}{h'(z)} = \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \frac{zg'(z)}{g(z)},$$

which readily shows that

$$\left| c |z|^{2\beta} + \left(1 - |z|^{2\beta} \right) \frac{zh''(z)}{\beta h'(z)} \right|$$

$$= \left| c |z|^{2\beta} + \left(1 - |z|^{2\beta} \right) \frac{1}{\beta} \left( \alpha \left( \frac{zf'(z)}{f(z)} - 1 \right) + \frac{zg'(z)}{g(z)} \right) \right|$$

$$\leq |c| + \frac{1}{|\beta|} \left( |\alpha| \left( \frac{z^2 f'(z)}{|f(z)|^2} \cdot \left| \frac{f(z)}{z} \right| + 1 \right) + \left| \frac{zg'(z)}{g(z)} \right| \right).$$

Furthermore, from the hypothesis $(2.1)$ of Theorem 2.1, we have

$$|f(z)| < M, \quad z \in \mathbb{U} \quad \text{and} \quad \left| \frac{zg'(z)}{g(z)} \right| \leq N, \quad z \in \mathbb{U}.$$ 

By applying the General Schwarz Lemma, we thus obtain

$$|f(z)| \leq M |z|, \quad z \in \mathbb{U}.$$
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Next, by making use of (2.6), we have

\[ \left| c |z|^{2\beta} + \left( 1 - |z|^{2\beta} \right) \frac{zh''(z)}{\beta h'(z)} \right| \]

\[ \leq |c| + \frac{1}{|\beta|} \left( |\alpha| \left( \left| \frac{z^2 f'(z)}{|f(z)|^2} \right| M + 1 + N \right) \right) \]

\[ \leq |c| + \frac{1}{|\beta|} \left( |\alpha| \left( \left( \frac{z^2 f'(z)}{|f(z)|^2} - 1 \right) - \frac{m-1}{2} |z|^{m+1} \right) M + \left( 1 + \frac{m-1}{2} |z|^{m+1} \right) M + 1 + N \right) \]

\[ \leq |c| + \frac{1}{|\beta|} \left( |\alpha| \left( \frac{m+1}{2} |z|^{m+1} M + \left( 1 + \frac{m-1}{2} |z|^{m+1} \right) M + 1 \right) + N \right) \]

\[ \leq |c| + \frac{1}{|\beta|} \left( |\alpha| \left( (m+1) M + 1 + N \right) \right) \]

\[ \leq |c| + \frac{1}{\Re(\beta)} \left( |\alpha| \left( (m+1) M + 1 + N \right) \right) \]

\[ \leq 1, \quad z \in \mathbb{U}, \]

where we have also used the hypothesis (2.2) of Theorem 2.1.

Finally, by applying Theorem 1.3, we conclude that the function \( F \); \( z \) defined by (1.2) is analytic and univalent in \( \mathbb{U} \). This evidently completes the proof of Theorem 2.1.

\[ \square \]

3. Applications of Theorem 2.1

First of all, upon setting \( m = 1 \) in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

**Corollary 3.1.** Let the function \( f \in A \) satisfy the condition (1.3) and suppose that \( M, N \) are real positive numbers, \( m > 0 \) and \( g \in \mathcal{P} \). Also let

\[ (3.1) \quad \Re(\beta) \geq |\alpha| (2M + 1) + N, \quad \alpha, \beta \in \mathbb{C}. \]

If

\[ (3.2) \quad |f(z)| < M, \quad z \in \mathbb{U}, \quad \left| \frac{zf'(z)}{g(z)} \right| \leq N, \quad z \in \mathbb{U} \]

and

\[ (3.3) \quad |c| \leq 1 - \frac{1}{\Re(\beta)} |\alpha| (2M + 1) - \frac{1}{\Re(\beta)} N, \quad c \in \mathbb{C}, \]

then the function \( F_{\alpha,\beta}(z) \) defined by (1.2) is analytic and univalent in \( \mathbb{U} \).

We next set

\[ g(z) = 1, \quad z \in \mathbb{U} \]

in Theorem 2.1, and thus obtain the following interesting consequence of Theorem 2.1.
Corollary 3.2. Let the function $f \in A$ satisfy the condition (1.3) and suppose that $M$ is a real positive number. Also let

$$\Re(\beta) \geq |\alpha|(m+1)M+1, \quad \alpha, \beta \in \mathbb{C}. \quad (3.4)$$

If

$$|f(z)| < M, \quad z \in U \quad (3.5)$$

and

$$|c| \leq 1 - \frac{1}{\Re(\beta)} |\alpha|(m+1)M+1, \quad c \in \mathbb{C}; \quad (3.6)$$

then the function

$$F_{\alpha,\beta}(z) = \left( \beta \int_{0}^{z} t^{\beta-\alpha-1}[f(t)]^{\alpha} dt \right)^{\frac{1}{\beta}}$$

is analytic and univalent in $U$.

Finally, upon setting

$$m = 1 \quad \text{and} \quad g(z) = 1, \quad z \in U$$

in Theorem 2.1, we obtain the following consequence of Theorem 2.1.

Corollary 3.3. Let the function $f \in A$ satisfy the condition (1.3) and suppose that $M$ is a real positive number. Also let

$$\Re(\beta) \geq |\alpha|(2M+1), \quad \alpha, \beta \in \mathbb{C}. \quad (3.7)$$

If

$$|f(z)| < M, \quad z \in U \quad (3.8)$$

and

$$|c| \leq 1 - \frac{1}{\Re(\beta)} |\alpha|(2M+1), \quad c \in \mathbb{C}; \quad (3.9)$$

then the function

$$F_{\alpha,\beta}(z) = \left( \beta \int_{0}^{z} t^{\beta-\alpha-1}[f(t)]^{\alpha} dt \right)^{\frac{1}{\beta}}$$

is analytic and univalent in $U$.

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